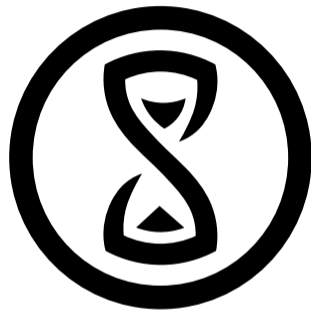


Galois Fields

Another view on bytes

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Stratum 0



Examples

- \mathbb{Q}
- \mathbb{R}
- \mathbb{C}

Axioms



associative in $+$	$a + (b + c) = (a + b) + c$
associative in \cdot	$a \cdot (b \cdot c) = (a \cdot b) \cdot c$
commutative in $+$	$a + b = b + a$
commutative in \cdot	$a \cdot b = b \cdot a$
identity of $+$	$a + 0 = a$
identity of \cdot	$a \cdot 1 = a$
	$0 \neq 1$
inverses of $+$	$\exists(-a) : a + (-a) = 0$
inverses of \cdot	$a \neq 0 \Rightarrow \exists(a^{-1}) : a \cdot (a^{-1}) = 1$
distributivity	$a \cdot (b + c) = (a \cdot b) + (a \cdot c)$



$$\mathbb{Z}_2 = \text{GF}(2)$$

- $\{0, 1\}$
- $0 + 0 = 0$
- $0 + 1 = 1$
- $1 + 1 = 0$
- $0 \cdot 0 = 0$
- $0 \cdot 1 = 0$
- $1 \cdot 1 = 1$



$$\mathbb{Z}_p = \text{GF}(p)$$

- $\{0, 1, \dots, p-1\}$
- $a + b = (a +_{\mathbb{R}} b) \bmod_{\mathbb{R}} p$
- $a \cdot b = (a \cdot_{\mathbb{R}} b) \bmod_{\mathbb{R}} p$



\mathbb{Z}_{256} ?

- $16 \cdot 16 = (16 \cdot_{\mathbb{R}} 16) \bmod_{\mathbb{R}} 256 = 256 \bmod_{\mathbb{R}} 256 = 0$
- but then... $0 = 0 \cdot (16^{-1}) = 16$.

This is broken (because 256 is not prime).



$GF(p^n)$

- polynomials of degree $< n$ over $GF(p)$.
- modulo some fixed irreducible polynomial R of degree n over $GF(p)$
- all finite fields of equal size are isomorphic, R doesn't matter



Example: $GF(3^2)$

- Choose $R = (x^2 + 1)$
- $(x + 1) + (2x + 0) = ((1 +_{\mathbb{R}} 2)x + (1 +_{\mathbb{R}} 0)) = (0x + 1) = 1$
- $(x + 1) \cdot (x + 1) = (x^2 + 2x + 1) \bmod (x^2 + 1) = (2x)$
- $(x + 1) \cdot (2x + 1) = (2x^2 + ((1 +_{\mathbb{R}} 2) \bmod_{\mathbb{R}} 3)x + 1) \bmod (x^2 + 1) = (2x^2 + 1) \bmod (x^2 + 1) = (2x^2 + 1) - (2x^2 + 2) = (-1) \bmod_{\mathbb{R}} 3 = 2.$



Byte-Sized Fields

$GF(2^8)$

- It has 256 elements
- We can represent the 7-degree polynomials with a bit field

$$0 = 0$$

$$x^4 + x + 1 = 0x13$$

- $a + b$ is binary-XOR (and $(-a) = a$).
- $a \cdot b$ is ... complicated



Byte-Sized Fields

Multiplication

- $a \cdot b = (a \cdot_{\mathbb{R}} b) \bmod_{\mathbb{R}} R$
- Straightforward: Polynomial multiplication + polynomial division
- Finite fields have generators, such that $a \neq 0 \Rightarrow \exists n : a = g^n$.
- $a \cdot b = g^{\log_g a} \cdot g^{\log_g b} = g^{\log_g a + \log_g b}$
- Store g^a and \log_a in lookup tables.
- Multiply becomes: zero-check; lookup; add; modulo (because $g^{256} = g^1$); lookup



Example

- AES field, $R = (x^8 + x^4 + x^3 + x + 1)$, suitable $g = 3$
- $0x15 + 0x05 = 0x10$
- $0x15 \cdot 0x05 = 3^{141} \cdot 3^2 = 3^{141+2} = 3^{143} = 65 = 0x41$



Applications

Our bytes now have interesting arithmetic properties!

Error Correction

- Reed-Solomon Codes, QR-codes
- Checksum via residues after division by polynomials over $GF(2^8)$.

Cryptography

- AES mix-columns step is matrix multiplication with scalars in $GF(2^8)$.

Bonus: Both use different R for the construction, thus separate log-tables.

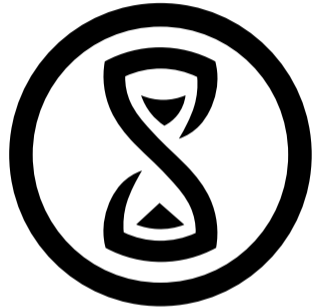
Questions?

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